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Escape to Infinity

Jon Perez Laraudogoitia[1]

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This paper shows how solutions to the equations of Newtonian mechanics, which become unbounded in a finite time, may be obtained for the case of rigid bodies of an arbitrary size subject to mutual elastic collisions alone, without any gravitational interaction. The absence of gravitation makes it possible to obtain by a new procedure a sort of singularity similar to those found for the *n*-body problem over the past 20 years.

The solutions found for the equations in dynamics which, for initial conditions given at t_0, become unbounded in a finite time are interesting for at least two reasons: (a) they constitute cases of singular solutions, that is, solutions defined analytically on some maximal interval $[t_0, t_1)$, with $t_1 < \infty$; and (b) they lead to a peculiar form of nondeterministic evolution of dynamic systems, since if $\lim_{t \to t_1} \left| (q_i(t) \right| = +\infty$, so that the system particles escape to infinity in a finite time, the temporal inversion of this process entails the unpredictable and spontaneous appearance of particles coming from spatial infinity.

A solution which becomes unbounded in a finite time naturally requires that the velocity (and therefore the kinetic energy) of the particles involved also grows in an unbounded way in that time. Mather and McGehee (1975) showed how this can happen by considering point particles subject to their mutual gravitational interaction and likewise subject to elastic binary collisions between some of them. More specifically, they used a system with four point particles in unidimensional movement, and they fixed initial conditions so that the distance $\left| \mathbf{r}_i - \mathbf{r}_j \right|$ between two particular particles tended to zero for $t \to t_1$. In such a case their potential gravitatory energy of interaction $-Gm_im_j/\left| \mathbf{r}_i - \mathbf{r}_j \right|$ tends to $-\infty$ for $t \to t_1$ and, as the system is conservative,

[1] Departamento de Logica y Filosofia de la Ciencia Universidad del Pais Vasco, 01006 Vitoria Gasteiz, Spain.

the kinetic energy of the particles involved tends to $+\infty$ for $t \to t_1$. In this way they obtain the result that the particles disappear into spatial infinity by $t = t_1$. Based on the same idea, Gerver (1984) made a justified conjecture that an unbounded solution would also be viable for a case with just gravitational interaction and no collision. The proof, however, that his model with five point particles, but this time in three-dimensional movement, leads to a singularity was only obtained several years later by Xia (1992). In line with these results, it is an interesting question whether solutions which become unbounded in a finite time are also possible in the case of particles which interact only through elastic collisions, in the absence of gravitation. In this paper we will show that this is indeed so. Moreover, our demonstration is interesting for another reason: it makes use of mathematical resources which are completely different from the ones employed in their seminal works by Mather and McGehee, Gerver, or Xia.

Since a solution which becomes unbounded in a finite time also requires unbounded kinetic energy, and since in our nongravitational model of elastic collisions all energy is kinetic and, in addition, it is preserved in every collision, it is obvious that what will be required is a system of particles whose initial kinetic energy is already infinite. If we make the (reasonable) requirement that the mass and the velocity of a particle always be finite, the preceding implies that we should start with an infinite set of particles each of which has a finite kinetic energy. The key to the construction of an unbounded solution will then lie in those particles reallocating the total amount of energy available through elastic collisions between themselves, so that the kinetic energy of each may grow unboundedly in a finite time. This will be the way for the escape to infinity. Thus our model requires an infinite number of particles, which is a theoretical idealization (needed to have an unlimited source of kinetic energy), but it must be remembered that the gravitational models of Mather and McGehee or Gerver and Xia require strictly point particles (which is another idealization, needed to have available an unlimited source of kinetic energy at the expense of the gravitational field). As there is no gravitational field in our model, the particles may be of any finite size whatsoever.

Consider an infinite set of rigid spheres $e_1, e_2, e_3, \ldots$ all of them having the same inertial mass m, but of arbitrary finite sizes, so that their respective diameters are $d_1, d_2, d_3, \ldots$. Suppose that all of them have their geometrical centers on the horizontal axis X and that at $t = 0$ they are moving along it in a positive direction in such a way that the velocity of e_i is $v_i = v(e_i) = i^2$, that e_{i+1} is situated to the left of e_i, between e_i and e_{i+2}, and that the distance between e_i and e_{i+1} is $d(e_i, e_{i+1}) = 2 \cdot i$ (taking the distance between two bodies A and B to be the smallest of the distances between a point in A and a point in B). Given those conditions, it is known that, according to the conservation laws, every time a binary collision takes place the colliding particles e_i and e_{i+1} will exchange velocity. Thus, even though initially the particle e_j is the one with velocity $v_j = j^2$, later on this velocity can be transmitted to different particles, so in general the expression "the particle having velocity $v_n = n^2$" will not always be used to refer to the same particle, but it will refer to that particle which (at the relevant instant) travels at the velocity indicated. Moreover, it is evident that

$$\begin{aligned} d(e_1, e_n) &= d(e_1, e_2) + d_2 + \ldots + d_{n-1} + d(e_{n-1}, e_n) \\ &= 2 \cdot 1 + d_2 + 2 \cdot 2 + \ldots + d_{n-2} + 2(n-2) + d_{n-1} + 2(n-1) \\ &= 2\left[\frac{1 + (n-1)}{2}\right](n-1) + \sum_{i=2}^{n-1} d_i \\ &= n^2 - n + \sum_{i=2}^{n-1} d_i \end{aligned}$$

Now, if we wished to determine, for example, at which instant after $t = 0$ the particle having velocity $v_n = n^2$ will collide with the particle having velocity $v_1 = 1$, it is obvious that the distance to be considered at $t = 0$ is not the physical distance $d(e_l, e_n)$, but a distance that we will henceforth call the "reduced distance" $d^*(e_1, e_n) = n^2 - n$. This reflects precisely the fact that intermediate collisions are instantaneous and that only an exchange of velocities takes place in them, so that it is not necessary to "cover" the space occupied by the $e_2, e_3, \ldots, e_{n-1}$, whose size is $\Sigma_{i=2}^{n-1} d_i$. As $d^*(e_l, e_i) = i^2 - i$ and $d^*(e_1, e_j) = j^2 - j$, assuming that $j > i$, then

$$\begin{aligned} d^*(e_i, e_j) &= d^*(e_l, e_j) - d^*(e_l, e_l) \\ &= j^2 - j - (i^2 - i) \\ &= (j - i)(j + i - 1) \end{aligned}$$

Finally, we will use the expression "effective distance traveled by the particle having velocity v" (in a certain interval of time Δt) to refer to the magnitude $v \cdot \Delta t$, that is, the distance obtained without computing the changes of location due only to the conversion of a particle having velocity v, e_α, into a particle having velocity v, e_β (with $\beta = \alpha \pm 1$). It is now easy to see that after $t = 0$ there will not be any multiple collisions in our model; all the collisions will be binary. Indeed, since with $j > i$ it holds that $d^*(e_i, e_j) = (j - i)(j + i - 1)$, the particle having velocity v_i will collide with the particle having velocity v_j when the former has traveled (after $t = 0$) an effective distance h_{ij}^* such that

$$\frac{h^*_{ij}}{i^2} = \frac{(j - i)(j + i - 1) + h^*_{ij}}{j^2}$$

(see Appendix), from which $h^*_{ij} = i^2(j + i - 1)/(j + i)$. Similarly, with $k = j + \alpha > j > i$ ($\alpha \geq 1$) it holds that $d^*(e_i, e_k) = (k - i)(k + i - 1)$ and the particle having velocity v_i will collide with the particle having velocity v_k when the former has traveled (after $t = 0$) an effective distance $h^*_{ik} = i^2(k + i - 1)/(k + i)$. Under such conditions, if a multiple collision (in other words, a simultaneous collision between more than two particles) took place, there would be at least three positive integers i, j, k with $i < j < k = j + \alpha$ such that $h^*_{ij} = h^*_{ik}$. But the condition $h^*_{ij} = h^*_{ik}$ implies $(j + i - 1)/(j + i) = (k + i - 1)/(k + i)$, that is $(j + i - 1)/(j + i) = (j + i - 1 + \alpha)/(j + i + \alpha)$, which is impossible if, as is the case, $\alpha > 0$. In our model we can conclude that every collision between particles will be a binary collision.

Suppose that $j > i$. At $t = 0$ the reduced distance between the particle having velocity v_i and the particle having velocity v_j is $d^*(e_i, e_j) = (j - i)(j + i - 1)$, given that at that instant the particle having velocity v_i is still e_i and the particle having velocity v_j is still e_j. Since their relative velocity is constant and equal to $v_j - v_i = j^2 - i^2$, if by (i, j) I denote the instant of time at which the collision between the particle having velocity v_i and the particle having velocity v_j takes place, it follows that

$$(i, j) = \frac{(j - i)(j + i - 1)}{(j^2 - i^2)} = \frac{j + i - 1}{j + i}$$

Obviously, if $i > j$, once again one can obtain

$$(i, j) = \frac{(i - j)(j + i - 1)}{(i^2 - j^2)} = \frac{j + i - 1}{j + i}$$

Therefore, for any distinct positive integers i, j, $(i, j) = (j + i - 1)/(j + i)$. If $k = j + \alpha > j$ (with α a positive integer) $(i, k) = (k + i - 1)/(k + i) = (j + i - 1 + \alpha)/(j + i + \alpha) > (j + i - 1)/(j + i) = (i, j)$. Therefore:

(I) For any distinct positive integers i, j, k with $j < k$, it holds that $(i, j) < (i, k)$, that is, the particle having velocity v_i will collide with the particle having velocity v_j before it does with the one having velocity v_k.

If by $[i, j]$ I denote the coordinates of the point on axis X at which the collision between the particles having velocity v_i and velocity v_j takes place, it follows straightaway from (I) and from the fact that all the particles are moving toward the region of increasing X coordinates that:

(II) For any distinct positive integers i, j, k with $j < k$ it holds that $[i, j] < [i, k]$.

In essence, this is all that is needed to show that each one of our particles e_i will escape to infinity in a finite time. We shall consider the case of e_1

first. From (I) it follows that the sequence of instants of time at which e_1 undergoes a collision is the infinite sequence

$$(S_1) \quad (1, 2) \quad (2, 3) \quad (3, 4) \quad (4, 5) \quad \ldots \quad (i, i+1) \quad \ldots$$

because at $t = 0$, e_1 is the particle having velocity v_1 and it successively becomes the particle having velocity v_2, the particle having velocity v_3, and so on. [Note that at $(j, j+1)$, e_1 becomes the particle having velocity v_{j+1}, and as $(j+1, k) \leq (j+1, j)$ for $k \leq j$, it follows that its next collision will take place at $(j+1, j+2)$. Thus it becomes the particle having velocity v_{j+2}.] As

$$(i, i+1) = \frac{i+1+i-1}{i+1+i} = \frac{2i}{2i+1}, \qquad \lim_{i\to\infty}(i, i+1) = 1$$

this means that at $t = 1$, e_1 will already have undergone all of the collisions in which it is involved (which, as we know, are infinite). Moreover, the interval of time $\Delta_{i,i+1}$ between the ith collision and the $(i+1)$th collision undergone by e_1 has the value

$$\Delta_{i,i+1} = \frac{2(i+1)}{2(i+1)-1} - \frac{2i}{2i+1} = \frac{2}{(2i+1)(2i+3)}$$

In this interval e_1 moves at velocity $v_{i+1} = (i+1)^2$, so the physical distance covered by e_1 in $\Delta_{i,i+1}$ is

$$D_i = \frac{2(i+1)^2}{(2i+1)(2i+3)} = \frac{2i^2+4i+2}{4i^2+8i+3}$$

Thus, the physical distance covered by e_1 from the moment at which it undergoes its first collision until $t = 1$ is given by the series $D_1 + D_2 + D_3 + \ldots = \lim_{n\to\infty} \Sigma_{m=1}^{n} D_m$. Since its general term $D_n = (2n^2 + 4n + 2)/(4n^2 + 8n + 3)$ is such that $\lim_{n\to\infty} D_n = 1/2 \neq 0$, it follows that the series $\Sigma_{n=1}^{\infty} D_n$ is not convergent. Since it is in addition a series of positive terms, it follows that $\lim_{n\to\infty} \Sigma_{m=1}^{n} D_m = +\infty$. This means that at $t = 1$ the particle e_1 has escaped into spatial infinity, and it has done so as a result of its successive collisions. Furthermore, as a consequence of this, and since by (II) the sequence of spatial coordinates at which e_1 undergoes a collision is the infinite sequence (chronologically ordered)

$$(S_2) \quad [1, 2] \quad [2, 3] \quad [3, 4] \quad [4, 5] \quad \ldots \quad [i, i+1] \quad \ldots$$

we can also conclude that sequence (S_2) is divergent, diverging to $+\infty$.

Consider now the case of particle e_k $(k \geq 2)$. From (I) it follows that the sequence of instants of time at which e_k undergoes a collision is the infinite sequence

(S$_3$) $(1, k)$ $(1, k+1)$ $(2, k+1)$ $(2, k+2)$ $(3, k+2)$ $(3, k+3)$ ... $(i, k+i-1)$ $(i, k+i)$ ...

since at $t = 0$, e_k is the particle having velocity v_k and it becomes, successively, the particle having velocity v_1, the particle having velocity v_{k+1}, the particle having velocity v_2, the particle having velocity v_{k+2}, etc. Notice, indeed, that in general at $(i, k+i-1)$, e_k becomes the particle having velocity v_i. Since for $k \leq j$, $(i, k) \leq (i, j)$, it follows that its next collision will take place at $(i, k+i)$. In this way, it becomes the particle having velocity v_{k+i}. The general term of (S$_3$) is of the form $(i, k+i-1)$ or of the form $(i, k+i)$. As

$$\lim_{i\to\infty}(i, k+i-1) = \lim_{i\to\infty}\frac{k+i-1+i-1}{k+i-1+i}$$

$$= \lim_{i\to\infty}\frac{2i+k-2}{2i+k-1} = 1 = \lim_{i\to\infty}(i, k+i)$$

$$= \lim_{i\to\infty}\frac{k+i+i-1}{k+i+i} = \lim_{i\to\infty}\frac{2i+k-1}{2i+k}$$

we conclude that at $t = 1$, e_k will already have undergone all of the collisions in which it is involved [and, given the infinite character of the sequence (S$_3$), we know there is an infinite number of them]. Furthermore, by (II), the sequence of spatial coordinates at which e_k undergoes a collision is the infinite sequence (chronologically ordered)

(S$_4$) $[1, k]$ $[1, k+1]$ $[2, k+1]$ $[2, k+2]$ $[3, k+2]$ $[3, k+3]$ ... $[i, k+i-1]$ $[i, k+i]$ ...

Let us now compare sequences (S$_2$) and (S$_4$). We have seen that (S$_2$) is divergent, in the specific sense that $\lim_{i\to\infty} [i, i+1] = +\infty$, and for that reason the following sequence of duplicates also diverges to $+\infty$:

(S$_2'$) $[1, 2]$ $[1, 2]$ $[2, 3]$ $[2, 3]$ $[3, 4]$ $[3, 4]$ ... $[i, i+1]$ $[i, i+1]$ $[i+1, i+2]$ ...

A way to specify the general term of (S$_2'$) is the following: for $n \geq 1$ the $(2n-1)$th term of (S$_2'$) is $[n, n+1]$ and the $(2n)$th is also $[n, n+1]$. A way to specify the general term of (S$_4$) is the following: for $n \geq 1$ the $(2n-1)$th term of (S$_4$) is $[n, k+n-1]$ and the $(2n)$th term is $[n, k+n]$. As $k \geq 2$, from (II) it follows that for all $n \geq 1$, $[n, n+1] < [n, n+k] = [n, k+n]$ and also $[n, n+1] \leq [n, n+k-1] = [n, k+n-1]$, namely, the terms in the sequence (S$_4$) are greater than or equal to the corresponding terms in the sequence (S$_2'$). Since (S$_2'$) diverges to $+\infty$, it automatically follows that (S$_4$) also diverges to $+\infty$. This means that at $t = 1$ particle e_k will have escaped to spatial infinity, and will have done so as a result of its successive collisions.

The above is enough to show that at $t = 1$ all the particles e_1, e_2, e_3, . . . have disappeared into infinity and, consequently, that at $t = 1$ a singularity takes place. It can be wondered whether, in our problem of infinite bodies evolving exclusively through elastic binary collisions, the set of initial conditions leading to the existence of a singularity is sufficiently "big," in a nontrivial sense (that is, in a sense that is independent of the mere possibility of describing temporal evolution in a different inertial frame of reference). Notice that our singularity was obtained solely from the following specifications: at $t = 0$, $d(e_i, e_{i+1}) = 2 \cdot i$ and $v_i = v(e_i) = i^2$. It becomes immediately apparent that the deduction leading to its existence can be reproduced step by step from new specifications so that there is a nonnumerable infinite set of initial conditions leading to the existence of singularities (differing from one another nontrivially) in the problem of infinite bodies subject to binary elastic collisions only. The question of whether this set is of measure zero or not remains open. The literature on singularities due to escape to infinity does not have anything definitive to say on this matter either. Only for the case of the planar four-body problem (four point particles under mutual gravitational interaction moving on the same plane) is it known (Saari, 1977) that the set of initial conditions which could potentially eventuate in a noncollision singularity has measure zero, but unfortunately, it is ignored whether in this case there will or will not actually be a noncollision singularity. Anosov (1985) suggested that the answer is positive, but offered no proof.

APPENDIX

The justification of the equation $h^*_{ij}/i^2 = [(j - i)(j + i - 1) + h^*_{ij}(1)/j^2$ is intuitively obvious, but it may be of some use to examine it in more detail for a better understanding of the difference between real physical displacement and effective distance traveled by the particle having velocity v. At $t = 0$ the physical distance between e_i and e_j is $d(e_i, e_j)$ and the reduced distance $d^*(e_i, e_j) = d(e_i, e_j) - \Sigma_{l=i+1}^{j-1} d_l$ (with the understanding that if $i = j - 1$, then $\Sigma_{l=i+1}^{j-1} d_l = 0$). When the particle having velocity v_i and the particle having velocity v_j collide, the former will have performed a real physical displacement h_{ij} (in a certain amount of time Δt) from its position at $t = 0$ and will by then have become a certain particle e_g. There are three possibilities:

(a) $g = i$. The effective distance traveled by the particle having velocity v_i (in Δt) will be $v_i \cdot \Delta t = h^*_{ij} = h_{ij}$. Therefore, from $t = 0$ until the instant it collides with the particle having velocity v_i, the particle having velocity

v_j must perform a real physical displacement $d(e_i, e_j) + h_{ij}$ and travel the effective distance

$$v_j \cdot \Delta t = d(e_i, e_j) + h_{ij} - \sum_{l=i+1}^{j-1} d_l$$

$$= d^*(e_i, e_j) + h_{ij}$$

$$= d^*(e_i, e_j) + h^*_{ij}$$

$$= (j - i)(j + i - 1) + h^*_{ij}$$

(b) $g < i$. The effective distance traveled by the particle having velocity v_i (in Δt) will be $v_i \cdot \Delta t = h^*_{ij} = h_{ij} - \Sigma_{l=g+1}^{i-1} d_l$ (with the understanding that if $g = i - 1$, then $\Sigma_{l=g+1}^{i-1} d_l = 0$). Therefore, from $t = 0$ until the instant it collides with the particle having velocity v_i, the particle having velocity v_j must perform a real physical displacement $d(e_i, e_j) + d_i + h_{ij}$ and travel an effective distance

$$v_j \cdot \Delta t = d(e_i, e_j) + d_{li} + h_{ij} - \sum_{l=g+1}^{j-1} d_l$$

$$= d(e_i, e_j) - \sum_{l=i+1}^{j-1} d_l + d_i - d_i + h_{ij} - \sum_{l=g+1}^{i-1} d_l$$

$$= d^*(e_i, e_j) + h^*_{ij}$$

$$= (j - i)(j + i - 1) + h^*_{ij}$$

(c) $g > i$. The effective distance traveled by the particle having velocity v_i (in Δt) will be $v_i \cdot \Delta t = h^*_{ij}$ such that $h_{ij} = v_i \cdot \Delta t - \Sigma_{l=i+1}^{g-1} d_l$ (again, if $i = g - 1$, we shall take it that $\Sigma_{l=i+1}^{g-1} d_l = 0$). Notice that it could be the case that $v_i \cdot \Delta t < \Sigma_{l=i+1}^{g-1} d_l$, that is, that the real physical displacement h_{ij} performed by the particle having velocity v_i is negative (in spite of the fact that $v_i = i^2 > 0$). This is due to the fact that the particle having velocity v_i at $t = 0$, namely e_i, becomes, successively, $e_{i+1}, e_{i+2}, \ldots, e_{g-1}, e_g$ during the interval of time Δt. Nonetheless, h^*_{ij} is always positive and has the value $h^*_{ij} = v_i \cdot \Delta t = h_{ij} + \Sigma_{l=i+1}^{g-1} d_l$. From $t = 0$ until the instant that it collides with the particle having velocity v_i, the particle having velocity v_j must perform a real physical displacement $d(e_i, e_j) + h_{ij}$ and travel the effective distance

$$v_j \cdot \Delta t = d\,(e_i, e_j) + h_{ij} - d_g - \sum_{l=g+1}^{j-1} d_l$$

$$= d(e_i, e_j) + h^*_{ij} - \sum_{l=i+1}^{g-1} d_l - d_g - \sum_{l=g+1}^{j-1} d_l$$

$$= d(e_i, e_j) - \sum_{l=i+1}^{j-1} d_l + h_{ij}^*$$

$$= d^*(e_i, e_j) + h_{ij}^*$$

$$= (j - i)(j + i - 1) + h_{ij}^*$$

Therefore, during the interval Δt between $t = 0$ and the time at which the particle having velocity v_i collides with the particle having velocity v_j, the former has traveled an effective distance $h_{ij}^* = v_i \cdot \Delta t$ and the latter an effective distance $(j - i)(j + i - 1) + h_{ij}^* = v_j \cdot \Delta t$. From these two equations we obtain the result $h_{ij}^*/v_i = [(j - i)(j + i - 1) + h_{ij}^*]/v_j$, and as $v_i = i^2$ and $v_j = j^2$ the equation that is the object of this appendix is proved.

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